Chapter 1

Functions and Function Notation

What is a Function?

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Function

A rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say "the output is a function of the input."

A *function* relates each element of a set with exactly one element of another set (possibly the same set). Let a be an element in set A and b an element in set B: if b is assigned to a, this is written as $f(a) = b$. If f is a function from A to B, we write $f: A \rightarrow B$

For example, for the function $f(x) = x^2, f(3) = 9, f(5) = 25, f(9) = 81$ etc.

The input to this function, is called the ***domain***. The output of this function, is called the ***codomain***. If $f(a) = b$, we say that b is the ***image*** of a and a is a ***pre-image*** of b. The ***range*** of f is the set of all images of elements of A.

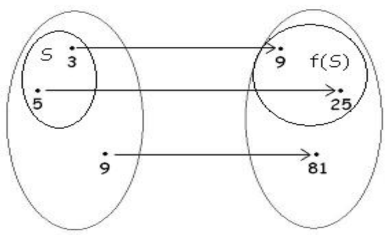
What's the difference between range and codomain?

Real-valued functions with the same domain can be added and multiplied:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

What do you think is the domain and codomain of $(f_1 + f_2)$ *and* $f_1 f_2$*?*

If S is a subset of A, the ***image*** of S is the *subset* of B that consists of the images of the elements of S. Then $f(S)$ is called the image of S.



One-to-one functions

One-to-one functions called also *injective* functions. A function $f: X \to Y$ is an injection if and only if $f(x) = f(y)$ implies that $x = y$ for all x and y in the domain of f. Check $f(x) = x + 1$.

Increasing/Decreasing functions

A function *f* is strictly increasing if $f(x) < f(y)$ whenever $x < y$ and *x* and *y* are in the domain of *f*.

Onto functions

Onto functions also called *surjective* functions. A function $f: X \to Y$ is a surjection if and only if for every element x of X there is an element y of Y with $f(x) = y$.
Check $f(x) = x^2$.

One-to-one Correspondence

A function is a one-to-one correspondence, or a *bijection*, if it is both one-to-one and onto.

Inverse function

Inverse function assigns to an element *b* belonging to B the unique element *a* in A such that $f(a) = b$. It is denoted by f^{-1}. So, $f^{-1}(b) = a$ when $f(a) = b$.

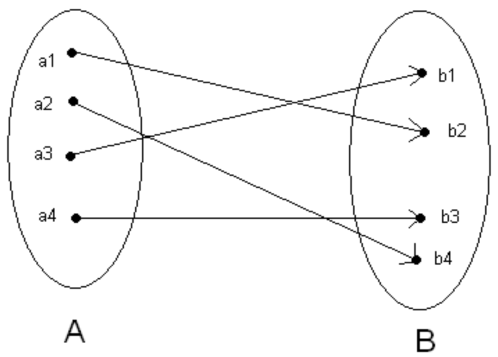
What is the inverse function of $f(x) = x^2$?

Composite functions

The *composition* of two functions *f* and *g* is denoted by $f \circ g$, and defined by $(f \circ g)(a) = f(g(a))$

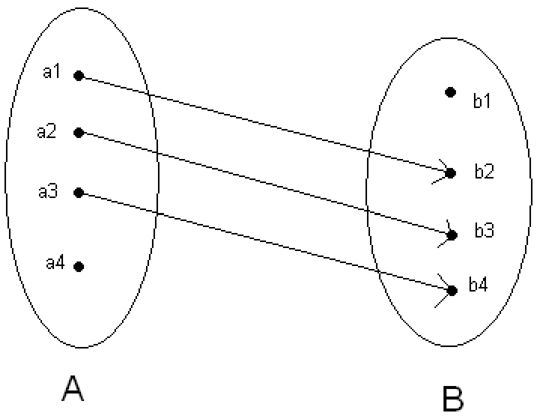
Exercises

1) For the following functions, state the i) domain ii) codomain iii) image iv) range
a)



b) $f: \mathbb{Z} \to \mathbb{Z}, f(x) = x + 1.$

c)



d) $f: \mathbb{N} \to \mathbb{N},\ f(x) = 100/x$.

Different types of functions

Constant functions

It is a function has the form $f(x) = c$, where c is a constant. For example, $f(x) = -5$ and $f(x) = \pi/2$.

Polynomials

The polynomials are functions on the form

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_\circ$

$a_n, a_{n-1}, a_{n-2}, \ldots, a_\circ$ $a_n \neq 0$ are real numbers. In this case n is the degree of the polynomial. For instance,

$$f(x) = 4x^5 - 3x^4 + 5x^2 + 16$$

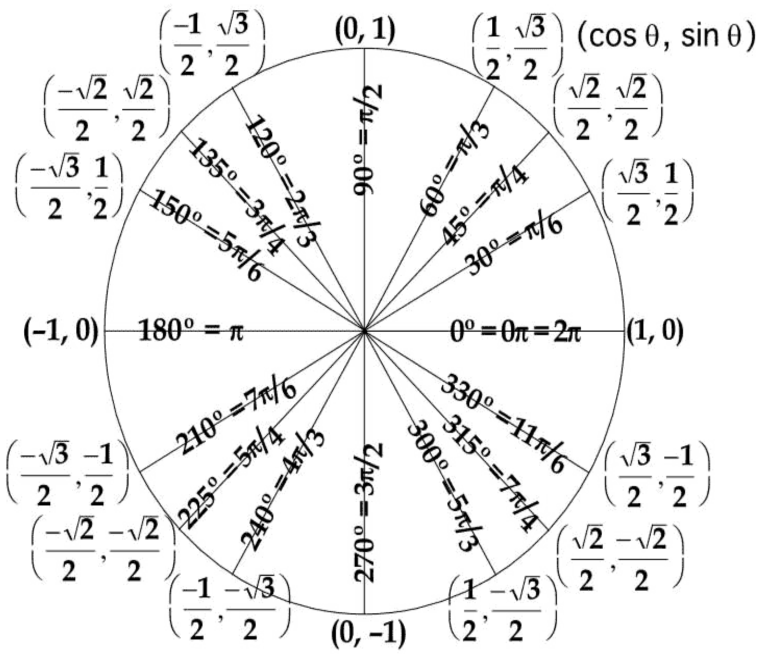
Is a polynomial function of degree 5 and the coefficients are 4,-3,5 and 16.

Rational functions

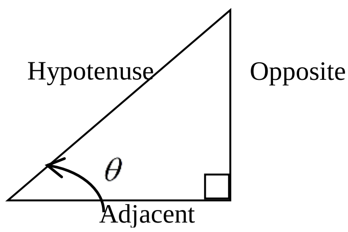
Are those the functions on the form

$f(x) = \dfrac{R(x)}{T(x)}$. For example, $f(x) = \dfrac{x^3 + 1}{x + 4}$.

The Trigonometric Functions



Right triangle definitions, where $0 < \theta < \dfrac{\pi}{2}$

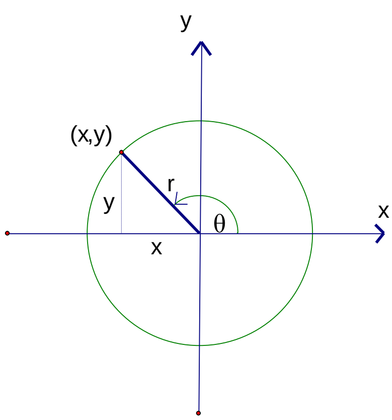


$$\sin\theta = \frac{opp.}{hyp.} \qquad \csc\theta = \frac{hyp.}{opp.}$$

$$\cos\theta = \frac{adj.}{hyp.} \qquad \sec\theta = \frac{hyp.}{adj.}$$

$$\tan\theta = \frac{opp.}{adj.} \qquad \cot\theta = \frac{adj.}{opp.}$$

Circular function definitions, where θ is any angle.



$r = \sqrt{x^2 + y^2}$,

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Reciprocal Identities

$$\sin u = \frac{1}{\csc u} \qquad \cos u = \frac{1}{\sec u} \qquad \tan u = \frac{1}{\cot u}$$

$$\csc u = \frac{1}{\sin u} \qquad \sec u = \frac{1}{\cos u} \qquad \cot u = \frac{1}{\tan u}$$

Negative Angle Identities

$\sin(-u) = -\sin u$ $\cos(-u) = \cos u$

$\csc(-u) = -\csc u$

$\sec(-u) = \sec u$

$\cot(-u) = -\cot u$

$\tan(-u) = -\tan u$

Tangent and Cotangent Identities

$$\tan u = \frac{\sin u}{\cos u} \qquad \cot u = \frac{\cos u}{\sin u}$$

Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1$$
$$1 + \tan^2 u = \sec^2 u$$
$$1 + \cot^2 u = \csc^2 u$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \cos\left(\frac{\pi}{2} - u\right) = \sin u$$
$$\csc\left(\frac{\pi}{2} - u\right) = \sec u \qquad \sec\left(\frac{\pi}{2} - u\right) = \csc u$$
$$\tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u$$

Sum and Difference Formulas

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$
$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$
$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Half Angle Identities

$$\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos u}{2}}$$
$$\cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1+\cos u}{2}}$$
$$\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos u}{1+\cos u}}$$

The domain and range of the trigonometric functions are as follows

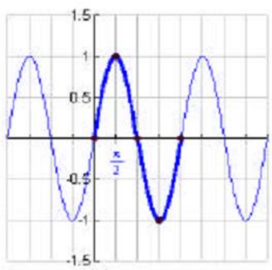
FUNCTION	DOMAIN	RANGE
$y = \sin x$	R	$[-1,1]$
$y = \cos x$	R	$[-1,1]$
$y = \tan x$	$R - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	R
$y = \cos ec x$	$R - \{n\pi : n \in I\}$	$R - (-1,1)$ or $(-\infty,1] \cup [1,\infty)$
$y = \sec x$	$R - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	$R - (-1,1)$ or $(-\infty,1] \cup [1,\infty)$
$y = \cot x$	$R - \{n\pi : n \in I\}$	R

$y = \sin x$

Domain:
All Reals

Range:
$[-1, 1]$

Period: 2π

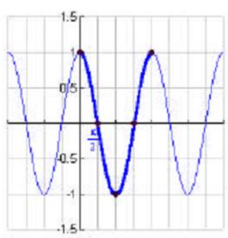


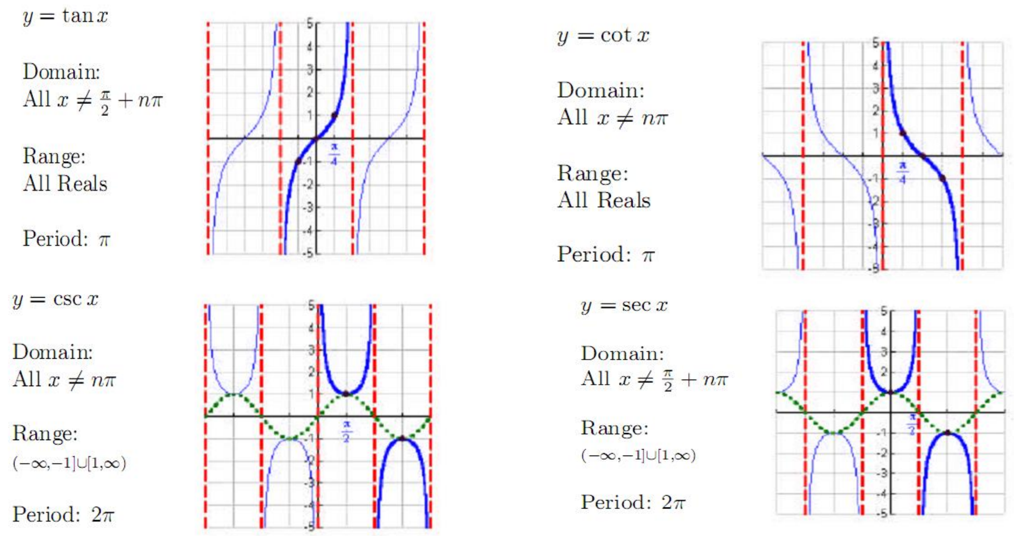
$y = \cos x$

Domain:
All Reals

Range:
$[-1, 1]$

Period: 2π

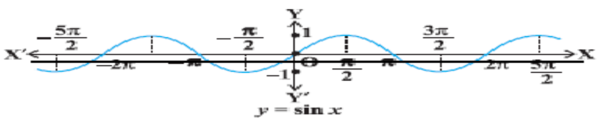




We know that a function is invertible iff it is one-one and onto

Now, observe the adjoining graph. The horizontal line below x-axis and parallel to x-axis intersects the graph of sine function at infinite many points. Therefore the sine function is not one-one.

Algebraically: $\sin\left(\frac{\pi}{2}\right)=\sin\left(\frac{5\pi}{2}\right)=\sin\left(\frac{9\pi}{2}\right)=\ldots=1$ i.e. there are infinite many angles in the domain of the function having same value 1. Thus the function can't have inverse at its existing domain.



Thus to over come this problem, we restrict the domain of sine function from the set R of real number to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ so that the function becomes one-one.

i.e. $\sin x:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[-1,1]$ becomes one-one and onto both thus the inverse of the function exists and we define it as

$\sin^{-1}x:[-1,1]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$$y=\sin x \Leftrightarrow x=\sin^{-1}x$$

Similarly other trigonometric functions are not one-one on their domains and hence be restricting their domains we can make them one-one and onto.

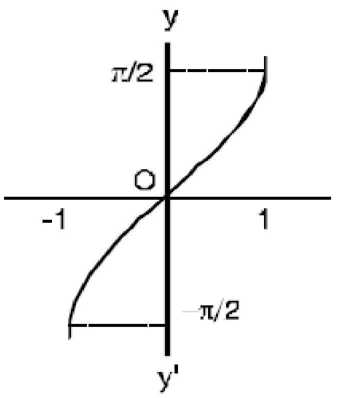
The domains and range of the inverse trigonometric are as follows

	DOMAIN		RANGE
$\sin^{-1}$:	$[-1, 1]$	$\rightarrow$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$:	$[-1, 1]$	$\rightarrow$	$[0, \pi]$
cosec^{-1} :	$\mathbf{R} - (-1, 1)$	$\rightarrow$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$:	$\mathbf{R} - (-1, 1)$	$\rightarrow$	$[0, \pi] - \{\frac{\pi}{2}\}$
$\tan^{-1}$:	$\mathbf{R}$	$\rightarrow$	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$:	$\mathbf{R}$	$\rightarrow$	$(0, \pi)$

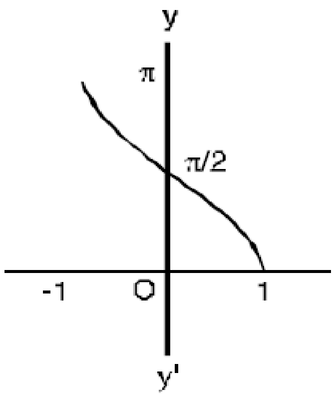
Note

1. $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.
2. Whenever no branch of an inverse trigonometric functions is mentioned, we mean the principal value branch of that function.
3. The value of an inverse trigonometric functions which lies in the range of principal branch is called the *principal value* of that inverse trigonometric functions.

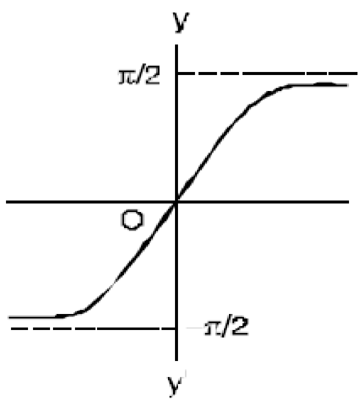
Graph of the inverse trigonometric function:



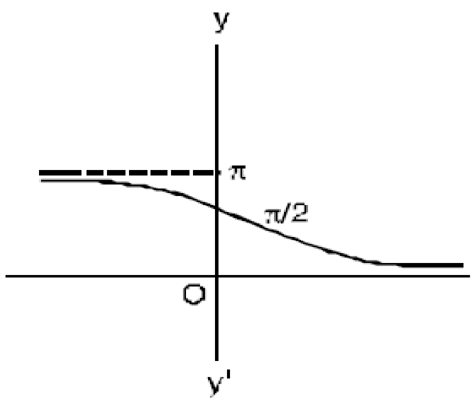
$y = \sin^{-1} x$



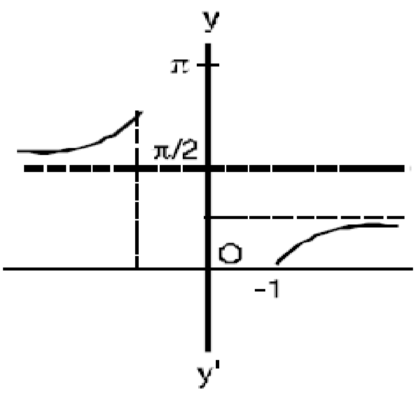
$y = \cos^{-1} x$



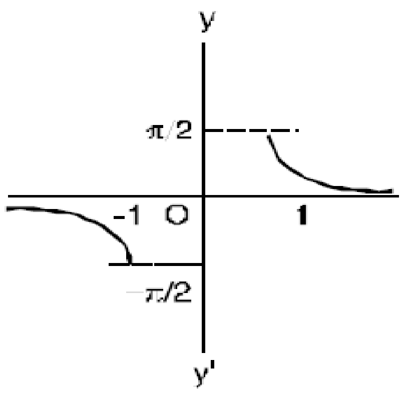
$y = \tan^{-1} x$



$y = \cot^{-1} x$



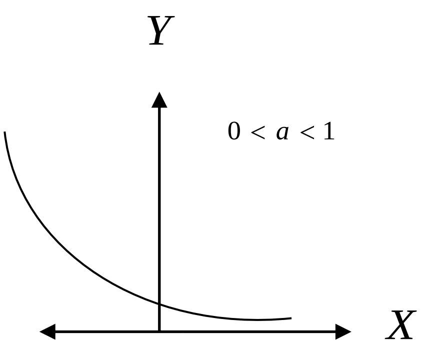
$y = \sec^{-1} x$

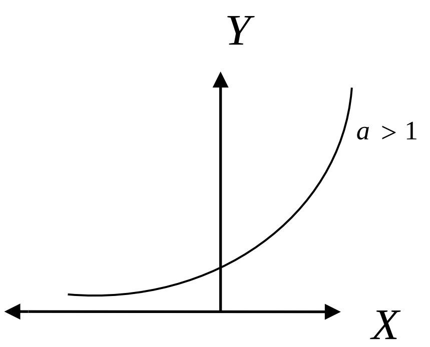


$y = \text{cosec}^{-1} x$

Exponential and Logarithmic Functions

The exponential function is a function taking the form $f(x) = a^x \quad a \neq 1,$





The exponential function has the following

1) $a^0 = 1$ 2) $a^{m+n} = a^m a^n$

3) $a^{m-n} = a^m / a^n$ 4) $(a^m)^n = a^{mn}$

When a is the natural base i.e, $a = e \cong 2.718$ then $f(x) = e^x$ is called the natural expansional function.

For all real numbers *a, m, n, p, x,* and *y,* where *x* > 0, a > 0, a ≠ 1.

1. If $a^x = a^y$, then $x = y$. — Property of Equality for Exponential Equations

2. $\log_a x = y$ *if and only if* $x = a^y$ — Definition of a Logarithm

3. $\log_{10} x = \log x$ — Common Logarithm (Base 10)

4. $\log_e x = \ln x$ — Natural Logarithm (Base *e*)

5. $\log_a a = 1$, $\log_a 1 = 0$, $\log_a a^x = x$, $a^{\log_a x} = x$ or 6. $\ln e = 1$, $\ln 1 = 0$, $\ln e^x = x$, $e^{\ln x} = x$ — Special Logarithms

7. $\log_a mn = \log_a m + \log_a n$ — Product Property

8. $\log_a \dfrac{m}{n} = \log_a m - \log_a n$ — Quotient Property

9. $\log_a m^p = p(\log_a m)$ — Power Property

10. $\log_a x = \dfrac{\log x}{\log a} = \dfrac{\ln x}{\ln a}$, $a \neq 1$ — Change of Base Formula

11. If $\log_a x = \log_a y$, then $x = y$. — Property of Equality for Logarithms

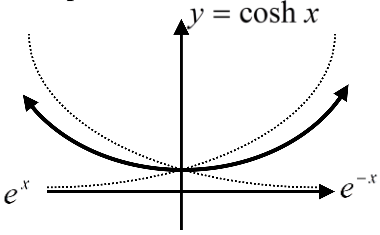
Hyperbolic Functions

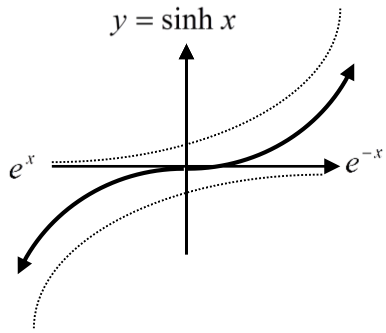
Definitions $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

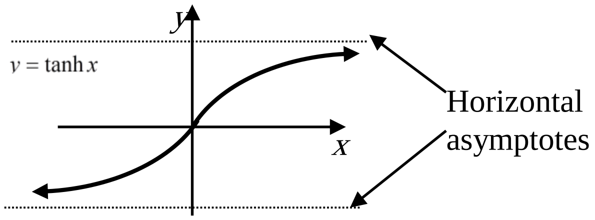
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \operatorname{cosech} x = \frac{1}{\sinh x}, \qquad \tanh x = \frac{1}{\coth x} = \frac{\sinh x}{\cosh x}$$

Graphs







Example Solve the equation $\cosh x = 2$

$$\tfrac{1}{2}\left(e^x + e^{-x}\right) = 2 \qquad \Rightarrow \quad e^{2x} - 4e^x + 1 = 0$$

$$\therefore \quad e^x = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3} \qquad \therefore \quad x = \ln\left(2 \pm \sqrt{3}\right)$$

$x = \ln\left(2 + \sqrt{3}\right)$ positive root and

$$x = \ln\left(2 - \sqrt{3}\right) = \ln\left(2 - \sqrt{3} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}\right) = \ln\frac{1}{\left(2+\sqrt{3}\right)} = -\ln\left(2+\sqrt{3}\right)$$

$$\cosh x + \sinh x = e^x$$
$$\cosh x - \sinh x = e^{-x}$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \operatorname{sec} h^2 x$$

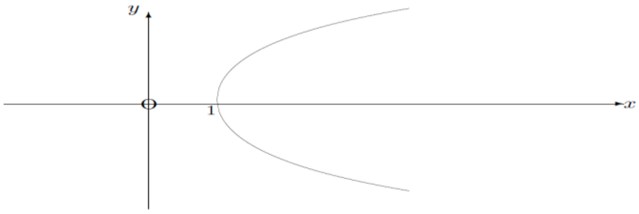
$$\coth^2 x - 1 = \operatorname{cos} ech^2 x$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$$

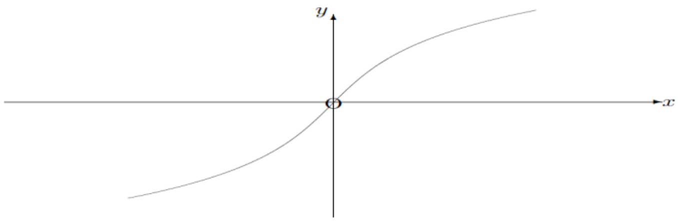
Inverse Hyperbolic Functions

Graphs

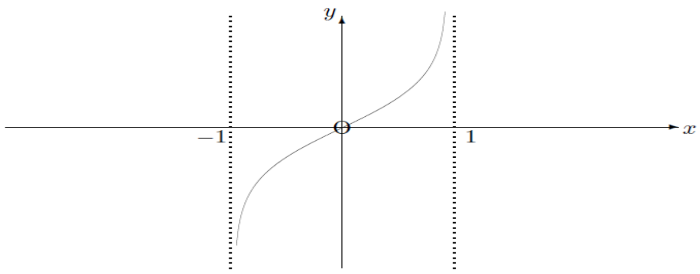
$y = \mathrm{Cosh}^{-1}x$



$y = \mathrm{Sinh}^{-1}x$



$y = \mathrm{Tanh}^{-1}x$



Log forms

$$\cosh^{-1} x = \ln\left\{x + \sqrt{x^2 - 1}\right\} \qquad x \geq 1$$

$$\sinh^{-1} x = \ln\left\{x + \sqrt{x^2 + 1}\right\} \qquad \text{all } x$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1$$

Be able to produce any of these

Example $\quad y = \tanh^{-1} x \quad \therefore \; x = \tanh y = \dfrac{e^{2y} - 1}{e^{2y} + 1}$

$$(e^{2y} + 1)x = e^{2y} - 1 \qquad \therefore \; e^{2y}(1 - x) = 1 + x \qquad \therefore \; e^{2y} = \frac{1 + x}{1 - x}$$

$$2y = \ln\left(\frac{1 + x}{1 - x}\right) \qquad \underline{y = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)}$$

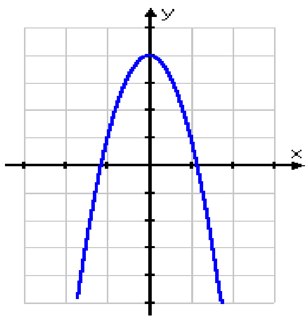
Even and odd functions

DEFINITION. A function **f** is **even** if the graph of **f** is symmetric with respect to the **y**-axis. Algebraically, **f** is **even** if and only if **f(-x) = f(x)** for all **x** in the domain of **f**.
A function **f** is **odd** if the graph of **f** is symmetric with respect to the origin. Algebraically, **f** is **odd** if and only if **f(-x) = -f(x)** for all **x** in the domain of **f**.

- Determine algebraically whether $f(x) = -3x^2 + 4$ is even, odd, or neither.

$f(-x) = -3(-x)^2 + 4$
$= -3(x^2) + 4$
$= -3x^2 + 4$

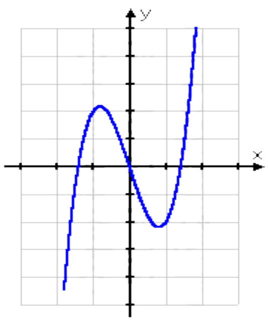
$f(x)$ is even



- Determine algebraically whether $f(x) = 2x^3 - 4x$ is even, odd, or neither.

$f(-x) = 2(-x)^3 - 4(-x)$
$= 2(-x^3) + 4x$
$= -2x^3 + 4x$

$f(x)$ is odd.



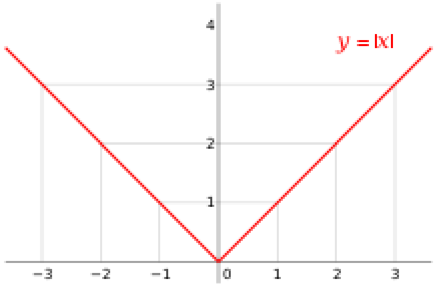
<u>Absolute or modulus functions</u>

For any real number x, the **absolute value** or **modulus** of x is denoted by $|x|$ (a vertical bar on each side of the quantity)

The absolute value of x is thus always either positive or zero, but never negative, since $x < 0$ implies $-x > 0$.

$$f(x) = |x| = \sqrt{x^2}$$

$$f(x) = |x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$



It is easy to see that the modulus function is even and has the following properties.

1) $|x+y| \le |x| + |y|$

2) $|x-y| \ge \big||x| - |y|\big|$

3) $|x\,y| = |x||y|$

4) $|x| \le a \Leftrightarrow -a \le x \le a$

5) $|x| \ge a \Leftrightarrow -a \ge x \quad or \quad x \ge a$

<u>Examples</u>

1) Prove that

$$\cos^{-1} a + \cos^{-1} b = \cos^{-1}\left\{ab - \sqrt{(1-a^2)(1-b^2)}\right\}.$$

<u>Solution</u>

Assume that $\quad x = \cos^{-1} a \quad , \quad y = \cos^{-1} b$

$\therefore a = \cos x \quad , \quad b = \sin y$

$$\therefore \cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$= ab - \sqrt{1-\cos^2 x}\sqrt{1-\cos^2 y}$$

$$= ab - \sqrt{1-a^2}\sqrt{1-b^2}$$

$$\therefore x + y = \cos^{-1} a + \cos^{-1} b = \cos^{-1}\left\{ab - \sqrt{1-a^2}\sqrt{1-b^2}\right\}$$

2) Let $g(x) = a + f(x)$, then prove that $g^{-1}(x) = f^{-1}(x-a)$.

solution

Assume that $y = g(x)$

$$y - a = f(x) \Rightarrow \therefore x = f^{-1}(y-a)$$

$$\therefore g^{-1}(y) = f^{-1}(y-a) \Rightarrow \therefore g^{-1}(x) = f^{-1}(x-a)$$

3) Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$.

Solution

Suppose that $\tan^{-1}\frac{1}{2} = x$, $\tan^{-1}\frac{1}{3} = y$

$\therefore \frac{1}{2} = \tan x$, $\frac{1}{3} = \tan y$

$$\because \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1 \Rightarrow x + y = \frac{\pi}{4}$$

4) Determine the domain of the following functions

$a)\quad f(x) = \frac{3}{\sqrt{x-5}} \qquad b)\quad y = \cos^{-1}(2x-5)$

Solutions

(a) The function $f(x) = \frac{3}{\sqrt{x-5}}$ is defined for x such that

$$x - 5 > 0$$

So $x > 5$ and hence $x \in]5, \infty[$.

(b) The function $y = \cos^{-1}(2x-5)$ is defined for all x such that

$$|2x-5| \le 1$$

Which means that

$$-1 \le 2x - 5 \le 1 \Rightarrow \therefore 2 \le x \le 3$$
$$x \in [2, 3]$$

5) Determine the values of x such that $|x-6| + |3x+2| \le 8$.

Solution

From the properties of the modulus function, we have

$|(x-6) + (3x+2)| \le |x-6| + |3x+2| \le 8$

$\therefore |4x-4| \le 8 \Rightarrow 4|x-1| \le 8 \Rightarrow |x-1| \le 2$

Then we get $-2 \le x-1 \le 2 \Rightarrow \therefore -1 \le x \le 3$

6) Check if the function $y = \dfrac{x^3}{2} + 1$ has inverse. If yes then find it.

Solution

It is easy to check that the function is one to one and onto. Now, we have

$\because y = \dfrac{x^3}{2} + 1 \Rightarrow \therefore x = \sqrt[3]{2(y-1)}$

Finally interchanging x with y we have

$$y = \sqrt[3]{2(x-1)}$$

and this is the inverse function of $y = \dfrac{x^3}{2} + 1$.

7) **Decide if** the following functions are even or odd

$a) \quad f(x) = x\sin x$

$b) \quad f(x) = x^3 - \sin x$

$c) \quad f(x) = 3\sin x + 2\cot x - x$

$d) \quad f(x) = \sqrt{2+x^2}$

Solutions

$a) \quad f(-x) = (-x)\sin(-x) = x\sin x = f(x)$

$\therefore$ f is even function

$b) \quad f(-x) = (-x)^3 - \sin(-x) = -(x^3 - \sin x) = -f(x)$

$\therefore$ f is odd function

$c) \quad f(-x) = 3\sin(-x) + 2\cot(-x) - (-x)$

$$= -3\sin x - 2\cot x + x = -f(x)$$

$\therefore$ f is odd function

$d) \quad f(-x) = \sqrt{2+(-x)^2} = \sqrt{2+x^2} = f(x)$

$\therefore$ f is even function.

Exercises

1) Decide if the following functions have inverse. If yes then find it.

$a)\quad f(x)=x^2+1$

$b)\quad f(x)=\dfrac{x-1}{x+1}\qquad x\neq -1$

$c)\quad f(x)=\cos^{-1}\dfrac{1}{\sqrt{x-3}}$

2) Decide if the following functions are even or odd.

$a)\quad f(x)=\ln\left[(x+\sin x)\cdot\cot 2x\right]$

$b)\quad f(x)=x^4+2x^2-4$

$c)\quad f(x)=x\cot x$

$d)\quad f(x)=x^3+\sin x-3\tan x$

3) Solve the following equations.

$a)\quad \tan^{-1}(1-x)+\tan^{-1}(1+x)=1/2$

$b)\quad \log(x^2+7)-2\log(x+1)=0$.